On Padé Approximation of Some Practical Functions

Javad Askari – Marnani [1] Parviz Jabedar – Maralani [2]

[1] Associate Professor, Department of Electrical Engineering Faculty of Electrical and Computer Engineering, Isfahan University of Technology, Isfahan, Iran
j-askari@cc.iut.ac.ir

[2] Professor, Department of Electrical Engineering, Faculty of Engineering,Tehran University, Tehran, Iran
pjabedar@ut.ac.ir

Abstract :

The Padé approximation of e^{Tz} has special importance in some engineering problems such as investigation of linear time-delayed systems. In this paper a novel method is introduced to construct Padé approximation form of some practical functions, such as e^{z}, $(1-z)^{\alpha}(1+z)^{\beta}$ with non-integers α and β, $\ln(1+z)$ and $(1+z)^{\beta}$ with non-integer β. This method gives a new closed form of Padé approximant of these functions by orthogonal polynomials. It introduces also a novel method to reduce the order of an ordinary differential equation. In other words, the method of this paper presents a new method to approximate the solution of an n-order ordinary differential equation by an $(n-1)$- order ordinary differential equation.

Keywords: Padé approximation, Difference calculus.

Submission date : 15 , Dec., 2012

Conditionally acceptance date: 2, June, 2014

Acceptance date : 31, Dec., 2014

Corresponding author: Javad Askari – Marnani

Corresponding author's address: Emam Khomini Str.,Faculty of Electrical and Computer , Engineering, Isfahan University of Technology, Isfahan, 84156-83111, Iran.

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1. Introduction

The Padé approximation of certain functions has many applications in the engineering problems, as well as numerical problems. For example time-delayed systems [1, 2], model order reduction [3,4], electromagnetic problems [5] and solution of Fredholm integral equation of the second kind [6] are some applications of Padé approximation. In such engineering problems, some practical functions such as Bessel's functions, $(1-z)^{\alpha}(1+z)^{\beta}$, e^{z} and $\ln(1+z)$ suggest itself. Among these functions Padé approximation of exponential function has an important role in investigation of linear timed-delayed systems. Padé approximation of exponential function is given by well-known relations

$$e^{z}M(z)-N(z)=O(z^{m+n+1}) \text{ as } z\to 0, \qquad (1)$$

$$M(z)=\sum_{k=0}^{m}\frac{(m+n-k)!m!}{k!(m-k)!}(-z)^{k}, \qquad (2)$$

$$N(z)=\sum_{k=0}^{n}\frac{(m+n-k)!n!}{k!(n-k)!}z^{k}, \qquad (3)$$

in which $O(z^{m+n+1})$ denotes Landau's symbol (see [6,7,8] for more details).

Padé approximation of some functions can be constructed using continued fractions method or determinant method [7-11]. Also, crucial role of orthogonality in Padé approximation is shown in some references [12, 13]. In order to unifying and to obtain a closed form, a novel method is given in this paper, to construct Padé approximation form of some practical functions, such as e^{z}, $(1-z)^{\alpha}(1+z)^{\beta}$ with non-integers α and β, $\ln(1+z)$ and $(1+z)^{\beta}$ with non-integer β (see [14] for an application of $\beta=0.5$). This method, which is based on difference calculus, gives a closed form of Padé approximant of these functions by orthogonal polynomials. It introduces also a method to reduce the order of an ordinary differential equation, which can be used in model order reduction problem. In fact, the approach of this paper presents a new method to approximate the solution of an n-order ordinary differential equation by an $(n-1)$- order one.

The paper is organized as follows. In section 2 some basic definitions are presented. In the third section, by means of a lemma and a theorem, a method is presented to construct a Padé approximation form of an analytical function. The relations (1)-(3) are re-obtained, extended and by the proposed method, a method is introduced to reduce the order of an ordinary differential equation in sub-section 3.1. To illustrate the proposed reduce order method, Bessel's equation and Airy's equation are considered in sub-section 3.1.2. Furthermore, the proposed method introduces a new closed form of Padé approximation of some practical functions such as $(1+z)^{\beta}$ for non-integer power β, e^{z}, $(1-z)^{\alpha}(1+z)^{\beta}$ and $\ln(1+z)$ in sub-section 3.2.

2. Definitions

Let $F(z)$ be an analytical function on a region Ω in the complex plane, and $M(z)$ and $N(z)$ be two polynomials with real coefficients and of degrees m and n, respectively. The rational function $R(z)=\frac{N(z)}{M(z)}$ is a Padé approximation of $F(z)$, if and only if, for a proper non-empty region $\Omega_c\subseteq\Omega$, $F(z)M(z)-N(z)=O(z^{m+n+1})$ for all $z\in\Omega_c$ [7]. Clearly, in this relation, $M(z)\neq 0$ and the rational function $R(z)$ is unique, although the polynomials $M(z)$ and $N(z)$ are not unique.

Let $j\le i$ be two non-negative integers and suppose $\binom{i}{j}$ denotes the combinatorial coefficient, i.e. $\binom{i}{j}=\frac{i!}{j!\,(i-j)!}$ in which $0!=1$ and $i!=1\times 2\times\cdots\times i$.

Definition 1: For two positive integers i and j, the function $A(i,j,t)$ is defined as follows.

$$A(i,j,t)=(-1)^{j}\,i!\,\frac{\Gamma(t)}{\Gamma(t-j)} \qquad (4)$$

in which t is a real number, $\Gamma(t)$ is Gamma function [15] and j is a positive integer.

Definition 2: The real numbers a_i and b_i are defined as follows:

$$a_i=\frac{1}{i!}\,E^{i}\,A(n,m,t)\Big|_{t=-n} \qquad (5)$$

$$b_i=\frac{1}{i!}\,\Delta^{i}\,A(n,m,t)\Big|_{t=-n} \qquad (6)$$

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where the shift operator E^i and forward difference operator $\Delta^i ; i=0,1,2,\cdots,$ are defined in usual manner [16], i.e.

$$E^0 f(t)=f(t) \quad \text{and for} \quad i\geq 1,$$
$$E^i f(t)=E^{i-1} f(t+1) \tag{7}$$
$$\Delta^0 f(t)=f(t) \quad \text{and for} \quad i\geq 1,$$
$$\Delta^i f(t)=\Delta^{i-1} f(t+1)-\Delta^{i-1} f(t) \tag{8}$$

Definition 3: Let $D^k=\dfrac{d^k}{dz^k}$ and Let $g(z)$ be a function which has a Taylor series for $z\in\Omega$. Using the shift operator E^i and forward difference operator Δ^i, we define the operators $g(zE)$ and $g(z\Delta)$ as follows.

$$g(zE)f(t)=\sum_{k=0}^{\infty}\frac{D^k g(0)}{k!}z^k E^k f(t)$$
$$=\sum_{k=0}^{\infty}\frac{D^k g(0)}{k!}z^k f(t+k) \tag{9}$$
$$g(z\Delta)f(t)=\sum_{k=0}^{\infty}\frac{D^k g(0)}{k!}z^k \Delta^k f(t) \tag{10}$$

3. A Padé approximation form

In this section we present a method by which, a new form of Padé approximant can be obtained for an analytical function. This form results in a closed form of Padé approximation of some practical functions such as e^z, $(1+z)^{\alpha}(1-z)^{\beta}$, $\ln(1+z)$, and $(1+z)^{\beta}$. We first prove a lemma which has an essential role in our method. Then, we summarize the main result of paper in Theorem1.

Lemma 1: Let a_i and b_i be defined by (5) and (6). Then

i. $a_i=0 ; n+1\leq i\leq n+m$ (11)

ii. $b_i=0 ; m+1\leq i$ (12)

iii. $a_i=\binom{n}{i}(n+m-i)! ; 0\leq i\leq n+m$ (13)

iv. $b_i=(-1)^i\binom{m}{i}(n+m-i)! ; 0\leq i\leq n+m$ (14)

v. $a_i=\sum_{k=0}^{i}\dfrac{b_k}{(i-k)!} ; 0\leq i\leq n+m$ (15a)

$$b_i=\sum_{k=0}^{i}\frac{(-1)^{i-k}a_k}{(i-k)!} ; 0\leq i\leq n+m \tag{15b}$$

Proof: According to (4), definition 2 and the property of the Gamma function, i.e. $\Gamma(t+1)=t\Gamma(t)$, the zeros of $A(n,m,t+i)$ are natural numbers in the interval $[1,m]$. Thus (11) is obtained for $1\leq i-n\leq m$. Also, since $A(n,m,t)$ is a polynomial of degree m, (12) is obvious. According to properties of combinatorial coefficient, we prove (13) for $0\leq i\leq n$. According to definitions (4) and (5),

$$a_i=\frac{(-1)^m\ n!}{i!}\prod_{k=1}^{m}(t+i-k)\Big|_{t=-n}$$
$$=\frac{n!}{i!}\prod_{k=1}^{m}(n-i+k)=\binom{n}{i}(n+m-i)!.$$

In a similar manner, (14) can be proved for $0\leq i\leq m$. Considering the relation $E^i=(\Delta+1)^i$ [16], (15a) and (15b) are resulted from (13) and (14). ■

Now, let $f(x)=A(n,m,x)$. For an analytical function $g(z)$ over a non-empty region Ω

$$g(zE)f(x)=\sum_{k=0}^{\infty}\frac{D^k g(0)}{k!}z^k E^k A(n,m,x)$$
$$g(zE)f(x)=\sum_{k=0}^{n}\frac{D^k g(0)}{k!}z^k E^k A(n,m,x)$$
$$+\sum_{k=n+1}^{n+m}\frac{D^k g(0)}{k!}z^k E^k A(n,m,x)$$
$$+\sum_{k=n+m+1}^{\infty}\frac{D^k g(0)}{k!}z^k E^k A(n,m,x).$$

Thus, according to Lemma 1

$$g(zE)f(x)\Big|_{x=-n}=\hat{N}(n,m,z)$$
$$+\sum_{i=0}^{\infty}\frac{D^{i+m+n+1}g(0)A(n,m,i+m+1)}{(i+n+m+1)!}z^{i+n+m+1}$$
$$=\hat{N}(n,m,z)+O(z^{m+n+1})$$

in which

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$$\hat{N}(n,m,z)=\sum_{k=0}^{n}\frac{D^k g(0)}{k!}A(n,m,-n+k)\,z^k .$$

On the other hand, using the relation $E=\Delta+1$ and Lemma 1, we have

$$g(zE)f(x)\big|_{x=-n}=$$

$$\sum_{k=0}^{\infty}\sum_{j=0}^{k}\frac{D^k g(0)}{k!}z^k\binom{k}{j}\Delta^j A(n,m,x)\big|_{x=-n}$$

$$=\sum_{j=0}^{\infty}\sum_{k=j}^{\infty}\frac{D^k g(0)}{j!(k-j)!}z^k\Delta^j A(n,m,x)\big|_{x=-n}$$

$$=\sum_{j=0}^{m}\sum_{k=0}^{\infty}\frac{D^{k+j} g(0)}{k!}z^k z^j\frac{1}{j!}\Delta^j A(n,m,x)\big|_{x=-n}$$

$$=\sum_{j=0}^{m}D^j g(z)\frac{z^j}{j!}\Delta^j A(n,m,x)\big|_{x=-n}=\hat{M}(n,m,z)$$

Hence, the relation

$$\hat{M}(n,m,z)-\hat{N}(n,m,z)=O(z^{n+m+1})$$

can be obtained. In other words;

$$\hat{N}(n,m,z)=\sum_{k=0}^{n}D^k g(0)\,a_k\,z^k \qquad (16a)$$

$$\hat{M}(n,m,z)=\sum_{j=0}^{m}D^j g(z)\,b_j\,z^j \qquad (16b)$$

$$\hat{\varepsilon}(n,m,z)=\hat{M}(n,m,z)-\hat{N}(n,m,z) \qquad (16c)$$

$$\hat{\varepsilon}(n,m,z)=\sum_{i=0}^{\infty}\frac{D^{i+m+n+1}g(0)A(n,m,i+m+1)}{(i+n+m+1)!}z^{i+n+m+1} \qquad (16d)$$

The region of convergence for (16d) can be obtained, for example, by D'Alembert's ratio test [17]. It can be seen that the region of convergence of $\hat{\varepsilon}(n,m,z)$ is the same as the region of convergence of the function $g(z)$. Also, some further manipulations show that

$$\hat{N}(n,m,z)=\int_0^{\infty}e^{-t}\,t^m(t+z\frac{d}{d\alpha})^n g(\alpha)\big|_{\alpha=0}\,dt \qquad (17a)$$

$$\hat{M}(n,m,z)=\int_0^{\infty}e^{-t}\,t^n(t-z\frac{d}{d\alpha})^m g(\alpha)\big|_{\alpha=0}\,dt \qquad (17b)$$

$$\hat{\varepsilon}(n,m,z)=\frac{z^{n+m+1}}{(n+m)!}\int_{t=0}^{1}(1-t)^{n+m}D^{m+n+1}g(ztE)E^{n+m+1}A(n,m,x)\big|_{x=-n}dt \qquad (17c)$$

These results are summarized in the following theorem.

Theorem 1: Let $g(z)$ be an analytical function over a region Ω containing the origin $0\in C$. Suppose for all $z\in\Omega$, the functions $\hat{M}(n,m,z)$, $\hat{N}(n,m,z)$ and $\hat{\varepsilon}(n,m,z)$ are defined by relations (16a)-(16c). Then for all $z\in\Omega$, (17a) and (17b) are satisfied and for a proper non-empty region $\Omega_c\subseteq\Omega$, $\hat{\varepsilon}(n,m,z)=O(z^{n+m+1})$ and $\hat{\varepsilon}(n,m,z)$ has power series (16d) and satisfies (17c). ■

The relations (16) are illustrated by the following examples given in next sub-sections.

3.1 Ordinary differential equations

Using relations (16), in this sub-section we present a novel method to reduce the order of an ordinary differential equation (ODE). This method, not only gives an approximate solution of an ODE, but also, it introduces a new method for model order reduction problem. We clarify this method by three examples in next subsections.

3.1.1: First order ODE

Suppose that $g(z)$ is a solution of the ODE $\frac{dy}{dz}=P(z)y+Q(z)$. For this function,

$$\frac{d^k y}{dz^k}=P_k(z)y+Q_k(z)\ \ (k\ge 0)$$

where $P_0(z)=1$, $P_1(z)=P(z)$, $Q_0(z)=0$, $Q_1(z)=Q(z)$ and

$$P_k(z)=e^{-I_1}\frac{d^k}{dz^k}(e^{I_1})\ (k\ge 0) \qquad (18a)$$

$$Q_{k+1}(z)=\frac{dQ_k(z)}{dz}+P_k(z)Q_1(z)\ (k\ge 0) \qquad (18b)$$

$$I_1=\int P_1(z)dz\,. \qquad (18c)$$

Thus, (16) result in the relation,

$$g(z)M_1(z)-N_1(z)=O(z^{m+n+1}),$$

in which

$$M_1(z)=\sum\nolimits_{j=0}^{j=m}P_j(z)b_j z^j \qquad (18d)$$

$$N_1(z)=\sum\nolimits_{i=0}^{i=n}D^i g(0)\,a_i\,z^i-\sum\nolimits_{i=0}^{i=m}Q_i(z)b_i z^i \qquad (18e)$$

For further illustration, suppose that the solution of the ODE $\frac{dy}{dz}=P(z)y+Q(z)$ with given initial

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condition $y(0)$, is $y(z)=u(z)v(z)$. For the functions $u(z)$ $(u(0)=1)$ and $v(z)$,

$$\frac{d^k u}{dz^k}=P_k(z)u \quad (k\geq 0), \tag{18f}$$

$$\frac{d^{k+1}v}{dz^{k+1}}=\frac{Q_k(z)}{u(z)} \quad (k\geq 0), \tag{18g}$$

in which $P_0(z)=1$, $Q_0(z)=Q(z)$, $P_k(z)$ satisfies (18a) and

$$Q_k(z)=e^{I_1}\frac{d^k}{dz^k}(e^{-I_1}Q(z)) \ (k\geq 0) \tag{18h}$$

Thus, the relations (16) result in the following relations,

$$u(z)M_u(z)-N_u(z)=O(z^{m+n+1}), \tag{18i}$$

$$b_0(v(z)-y(0))+\frac{1}{u(z)}\bar{M}_v(z)-\bar{N}_v(z)=O(z^{m+n+1}) \tag{18j}$$

in which $M_u(z)$ is given by (18d) and

$$N_u(z)=\sum_{j=0}^{j=n}P_j(0)a_jz^j,$$

$$\bar{M}_v(z)=\sum_{j=1}^{j=m}Q_{j-1}(z)b_jz^j,$$

$$\bar{N}_v(z)=\sum_{j=1}^{j=n}Q_{j-1}(0)a_jz^j.$$

3.1.2: Second order ODE

Suppose that $g(z)$ is a solution of the ODE $\frac{d^2y}{dz^2}=P(z)\frac{dy}{dz}+Q(z)y$. For this function,

$$\frac{d^{k+1}y}{dz^{k+1}}=P_k(z)\frac{dy}{dz}+Q_k(z)y \quad (k\geq -1)$$

where, $P_{-1}(z)=0, P_0(z)=1$, $P_1(z)=P(z)$, $Q_{-1}(z)=1, Q_0(z)=0$ and $Q_1(z)=Q(z)$. For $k\geq 0$, the function $Q_k(z)$ satisfies in (18b) and for $k\geq 0$

$$P_k(z)=e^{-I_1}\frac{d^k}{dz^k}(e^{I_1})+\sum_{i=1}^{i=k-1}e^{-I_1}\frac{d^{k-1-i}}{dz^{k-1-i}}(e^{I_1}Q_i), \tag{18k}$$

in which I_1 is given by (18c). Thus, (16) result in the following ODE,

$$\frac{dg(z)}{dz}M_1(z)+g(z)M_0(z)-\hat{N}(z)=O(z^{m+n+1}), \tag{18l}$$

in which $M_1(z)$ and $\hat{N}(z)$ are given by (18d) and (18e) and $M_0(z)=\sum_{j=0}^{j=m}Q_j(z)b_jz^j$.

For instance, for zero order Bessel function [15,18] and for $k\geq 2$,

$$Q_k(z)=\sum_{i=1}^{i=k-2}\frac{d^i}{dz^i}(-P_{k-i-1}),$$

$$P_k(z)=(-1)^k k!z^{-k}+\sum_{i=1}^{i=k-1}z\frac{d^{k-1-i}}{dz^{k-1-i}}(z^{-1}Q_i)$$

and $P_k(z^{-1})$ is a polynomial with degree k and $Q_k(z^{-1})$ is a polynomial with degree $k-1$. For Airy's equation [15], $P_k(z)=\sum_{i=1}^{i=k-1}\frac{d^{k-1-i}}{dz^{k-1-i}}Q_i$ is a polynomial with positive coefficients and degree $\frac{-3}{4}+\frac{3}{4}(-1)^{k+1}+\frac{k+1}{2}$ and $Q_k(z)$ is a polynomial with positive coefficients and degree $\frac{-3}{4}+\frac{3}{4}(-1)^k+\frac{k}{2}$.

3.2 Functions

In this sub-section, we present a new method to construct a Padé approximation form of an analytical function in a closed form. We clarify this method by considering some practical functions such as e^z, $(1+z)^\alpha(1-z)^\beta$, $\ln(1+z)$, and $(1+z)^\beta$ in next subsections.

3.2.1: Exponential function

For exponential function $g(z)=e^z$, (16) result in the following well known relation (for instance, see [19]),

$$e^zM(n,m,z)-N(n,m,z)=\varepsilon(n,m,z),$$

in which

$$N(n,m,z)=\sum_{i=0}^{n}a_iz^i=\int_0^\infty e^{-t}t^m(t+z)^n dt \tag{19}$$

$$M(n,m,z)=\sum_{i=0}^{m}b_iz^i=\int_0^\infty e^{-t}t^n(t-z)^m dt \tag{20}$$

$$\varepsilon(n,m,z)=(-1)^m z^{n+m+1}\int_{t=0}^{1}(1-t)^n t^m e^{zt}dt = \sum_{i=0}^{\infty}\frac{A(n,m,i+m+1)}{(i+m+n+1)!}z^{i+n+m+1}. \tag{21}$$

For the function $g(z)=z^me^{-z}$, (16) and associated Laguerre polynomials [15] result in another form of Padé approximation of exponential function as follows.

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$$e^{-z}\sum_{i=0}^{m}\binom{2m-i}{m}L_i^{m-i}(z)-1=O(z^{m+1}) \text{ as } z\to 0. \quad (22)$$

For the function $g(z)=e^{-z^2}$, (16) and Hermit polynomials [15] result in a form of Padé approximation of exponential function as follows.

$$P(z^2)=\sum_{i=0}^{m}(-1)^i b_i H_i(z) z^i \quad (23a)$$

$$Q(z^2)=\sum_{0\le 2k\le n}\frac{(2k)!}{k!}a_{2k}z^{2k} \quad (23b)$$

$$e^{-z}P(z)-Q(z)=O(z^{\frac{m+n+1}{2}}) \text{ as } z\to 0. \quad (23c)$$

By relations (16) and suitable functions $g(z)$ and $A(i,j,t)$, a Padé approximation form of exponential function which satisfies extra conditions can be obtained. For example, let α be a non-negative integer

and $A(i,j,t)=(-1)^j(t-1)^{\alpha}\, i!\,\frac{\Gamma(t)}{\Gamma(t-j)}$. Then, there are two polynomials $M_{ex}(z)$ and $N_{ex}(z)$ with degrees m and n respectively such that $e^z M_{ex}(z)-N_{ex}(z)=O(z^{m+n+1-\alpha})$ as $z\to 0$.

3.2.2: Function $(1+z)^{\beta}$

For the function $g(z)=(1+z)^{\beta+m}$ with non-integer power β, the relations (16) result in a form of Padé approximation of $(1+z)^{\beta}$ as follows.

$$N_{\beta}(n,m,z)=\sum_{k=0}^{n}\frac{\Gamma(\beta+m+1)}{\Gamma(\beta+m+1-k)}a_k z^k \quad (24a)$$

$$M_{\beta}(n,m,z)=\sum_{k=0}^{m}\frac{\Gamma(\beta+m+1)}{\Gamma(\beta+m+1-k)}b_k(1+z)^{m-k} z^k \quad (24b)$$

$$\varepsilon_{\beta}(n,m,z)=(1+z)^{\beta}M_{\beta}(n,m,z)-N_{\beta}(n,m,z) \quad (24c)$$

$$\varepsilon_{\beta}(n,m,z)=\sum_{i=0}^{\infty}\frac{\Gamma(\beta+m+1)}{\Gamma(\beta+m-i-n)}\frac{A(n,m,i+m+1)}{(i+n+m+1)!}z^{i+n+m+1} \quad (24d)$$

3.2.3: Function $\ln(1+z)$

For the function $g(z)=(1+z)^m\ln(1+z)$ and two positive integers $m<n$, (16) result in a form of Padé approximation of $\ln(1+z)$ as follows.

$$N_l(n,m,z)=\sum_{k=1}^{n}c_k a_k z^k+\sum_{k=1}^{m}c_k b_k(1+z)^{m-k}z^k \quad (25a)$$

$$c_i=i!\sum_{k=0}^{i-1}\binom{m}{k}\frac{(-1)^{i-k-1}}{i-k} \quad (25b)$$

$$M_l(n,m,z)=\sum_{k=0}^{m}\frac{m!}{(m-k)!}b_k(1+z)^{m-k}z^k \quad (25c)$$

$$\ln(1+z)M_l(n,m,z)-N_l(n,m,z)=O(z^{m+n+1}) \text{ as } z\to 0. \quad (25d)$$

3.2.4: Function $(1-z)^{\alpha}(1+z)^{\beta}$

For the function $g(z)=(1-z)^{m+n+\alpha}(1+z)^{m+n+\beta}$ with non-integers α and β and two positive integers m and n, (16) and Jacobi's Polynomials [15], result in a form of Padé approximation of $(1-z)^{\alpha}(1+z)^{\beta}$ as follows.

$$(1+z)^{\alpha}(1-z)^{\beta}M_{\alpha,\beta}(n,m,z)-N_{\alpha,\beta}(n,m,z)=O(z^{m+n+1}) \quad (26a)$$

$$M_{\alpha,\beta}(n,m,z)=\sum_{i=0}^{i=m}(-1)^i 2^i i!\,b_i(1-z^2)^{m+n-i}z^i P_i^{(m+n-i+\alpha),(m+n-i+\beta)}(z) \quad (26b)$$

$$N_{\alpha,\beta}(n,m,z)=\sum_{i=0}^{i=n}(-1)^i 2^i i!\,a_i P_i^{(m+n-i+\alpha),(m+n-i+\beta)}(0)\,z^i \quad (26c)$$

4. Conclusion

The Padé approximation is an important technique in some engineering problems and numerical methods. Among functions, Padé approximation of exponential function has an important role in electrical engineering problems. Analyzing time-delayed systems, or synthesis of a delay filter or methods which compute inverse Laplace transform numerically, are some applications of Padé approximation of exponential function.

In this paper, a novel method, which is based on difference calculus, is given to construct a Padé approximation form of some practical functions. This new method gives a closed form of Padé approximant of some practical functions by orthogonal polynomials. It introduces also a new method to reduce the order of

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model of a system described by an ordinary differential equation. The obtained results can be used in practical problems. For instance, based on the results of this paper and references [20, 21], our further research will concentrate on current-voltage modeling in doped carbon nanotube field effect transistors, and will concentrate on control of synchro-motors.

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