Bidding Strategy in Demand Response Exchange Market

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Abstract:

Demand response (DR) has many beneficiaries in the electricity market. There are independent players who are interested in DR, which include: transmission system owners, distributors, retailers, and aggregators. In this paper DR is introduced as a tradable commodity that can be exchanged between DR buyers and sellers in a pool-based market which is called demand response exchange (DRX). DRX operator (DRXO) collects DR offers and bids from the buyers and sellers. In this paper, a novel approach has been presented for buyers to bid in a DRX market. Also a dynamic approach has been proposed for DR sellers’ participation in DRX market. In the proposed approach, the buyers should forecast their loads and energy market prices. An ARIMA method is used for these forecasts. Then, a dynamic approach is proposed for DR sellers in order to maximize their profits. The proposed scheme is tested using Spain market data. The results show the efficiency and accuracy of the proposed approach.

Keywords: ARIMA method, Demand response exchange, DR buyers bidding, Dynamic participation.

Submission date: 07 Sep. 2012
Conditional acceptance date: 19 May 2013
Acceptance date: 08 July 2013

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1. Abbreviations and Acronyms

<table>
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<tr>
<th>Abbreviation</th>
<th>Description</th>
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<tbody>
<tr>
<td>DR</td>
<td>Demand Response</td>
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<tr>
<td>DRX</td>
<td>Demand Response Exchange</td>
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<td>DRXO</td>
<td>Demand Response Exchange Operator</td>
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<tr>
<td>DRB</td>
<td>Demand Response Buyer</td>
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<td>DRS</td>
<td>Demand Response Seller</td>
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<tr>
<td>TBRP</td>
<td>Time-Based Rate Program</td>
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<tr>
<td>IBP</td>
<td>Incentive-Based Program</td>
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<tr>
<td>MBP</td>
<td>Market-Based Program</td>
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<tr>
<td>ISO</td>
<td>Independent System Operator</td>
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<tr>
<td>TSO</td>
<td>Transmission System Owner</td>
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Notation

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
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<tbody>
<tr>
<td>i</td>
<td>Indicator for the number of customers</td>
</tr>
<tr>
<td>t</td>
<td>Indicator for the number of hours</td>
</tr>
<tr>
<td>l</td>
<td>Number of customers</td>
</tr>
<tr>
<td>T</td>
<td>Scheduling time horizon</td>
</tr>
<tr>
<td>p_t</td>
<td>Electricity price in hour t ($/MWh)</td>
</tr>
<tr>
<td>p_b</td>
<td>Electricity price in based hour</td>
</tr>
<tr>
<td>l_t</td>
<td>Amount of load in hour t</td>
</tr>
<tr>
<td>l_b</td>
<td>Amount of load in the base load interval</td>
</tr>
<tr>
<td>DRXCP</td>
<td>Demand response exchange clearing price [in $/MWh]</td>
</tr>
<tr>
<td>DR_i</td>
<td>Sold DR from i-th customer</td>
</tr>
<tr>
<td>RD</td>
<td>Required DR</td>
</tr>
<tr>
<td>p_t*</td>
<td>Actual price data in hour t</td>
</tr>
<tr>
<td>p_t#</td>
<td>Forecasted price data in hour t</td>
</tr>
<tr>
<td>l_t*</td>
<td>Actual load data in hour t</td>
</tr>
<tr>
<td>l_t#</td>
<td>Forecasted load data in hour t</td>
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2. Introduction

Power systems have been restructured and deregulated since 1990. As a result of restructuring, local utilities have been broken up into a number of independent players including: Generation Companies (Genco), Independent System Operator (ISO), Distribution Companies (Disco), retailers and aggregators [1]. In a competitive electricity market, Demand Response (DR) programs play an important role in improving market efficiency [2].

Transmission System Owners (TSO), distributors, retailers, and aggregators are independent players who are interested in DR benefits. DR enabling can improve reliability indices. A TSO can benefit from DR as a result to improve his/her network reliability [3], and distributors can manage network constraints at the distribution level by using of DR [4].

Retailers are exposed to financial risks due to market price volatility, because they purchase electricity from the wholesale market at volatile rates and sell it to consumers at a flat rate [5]. By reducing consumption during price spikes period, retailers may cover a part of these risks [6]. Reference [7], discusses about retailers bidding in order to determine the optimal demand curve for a retailer in electricity markets. Also, many researches have focused on price spike reduction by using of demand response [8] - [10].

DRPs are divided into three basic categories so-called, Time-Based Rate Programs (TBRPs), Incentive-Based Programs (IBPs) and Market-Based Programs (MBPs) as depicted in “Fig. 1” [11]. In TBRPs, the electricity price changes for different periods, so customers should adjust their consumption according to the time and associated tariffs. In IBPs, customers are being encouraged with independent system operator or local utility to moderate their consumption. In the market-based approach, all players are categorized in two groups: DR buyers (DRBs) as well as sellers (DRSs). DRBs need demand response to improve their business and system reliability while, DRs are aggregators and customers who sell DR to increase their benefit.

Aggregators negotiate the amount of combined DR of their consumers with TSO, distributors, and retailers. DR buyers want to improve the reliability of their own electricity-dependent businesses and systems. Sellers of DR have the capacity to significantly modify electricity demand. Recently, many researches have been introduced in demand response programs. Electricity price reduction, mitigating transmission network congestion, security enhancement, and improvement of market performance are the main aims of these researches [1], [12-16].

As introduced in [17], DR can be treated as a tradable commodity in a market which is completely separated from energy market. Market performance under demand response exchange market is better than conventional bilateral approaches [17]. Market total benefit is equal to the summation of the benefit of all players who are participated in DRX market and is equal to the confined area in “Fig. 2” between supply...
Fig. 2: DRX market-clearing model template

and demand curves that are sorted increasingly and decreasingly, respectively. The market total benefit is equal to the area depicted by (A+B) in “Fig. 2”. \( x^* \) and \( p^* \) in “Fig. 2” indicate the equilibrium point that is achieved from clearing the selling and purchasing curves.

DR is a resource which is integrated to improve the reliability of both network and market. In a DRX market, the demand response exchange operator (DRXO) collects both the aggregated demand and individualized supply curves. Then, it balances the supply and demand at a common price [17], [18].

This paper discusses a new concept of DR context called DRX. By defining the commercial model for DR, DR exchange procedure gain a new concept, in which DR is traded by DRXO in a pool-based market. In this study, a new approach is introduced for players’ participation in DRX market. Indeed, In this paper, instead of using constant curves that have been used in previous works [1, 17, 18], a novel procedure is proposed to determine the DR demand curves which are completely based on the forecasted load and price data in a horizon time. While in the previous studies, constant DR demand curves were considered, the main advantage of the proposed method in this paper is to obtain DR demand functions dynamically over the assumed period. In this approach, players should forecast the load and price variations. There are various approaches to forecast the load and price. ARIMA models [19], [20], wavelet transform model [21], [22], and another approaches are introduced for load forecasting. In [23], the authors introduce another way by using a hybrid method. In this paper, ARIMA method is used for load and price forecasting. Also, a linear bid/offer curve has been assumed for all players [24], [25]. Then, based on this model, a linearly decreasing demand curve for retailer participation in

Fig. 1: Classification of DR programs

DR

Price Based
Incentive Based
Market Based
Programs
Programs
Programs

Time of Use
Direct Load Control
Demand Bidding

Critical Peak Pricing
Interruptible/Curtailable
Ancillary Service
Programs
Emergency DR
Capacity Market

Real Time Pricing


DRX market will attain. Also, a dynamic method is proposed for DR sellers’ participation. Indeed, The main aim of this paper is to propose a dynamic procedure for determining the DR demand curves during the considered time horizon to maximize the DRBs’ profit which is based on the load and price data. This method determines the behavior of each buyer according to the other participants’ behavior in the past times and regarding the expected amount of required DR in each hour. This paper deals with DR concept as a commercial commodity which can be exchanged among the groups of buyers and sellers in a pool based market that is managed by DRXO.

The rest of the paper is organized as follows. In section 3, a novel model for participation of DR buyers in a DRX market is introduced. Section 4 introduces a dynamic approach for customers’ participation in DRX. Section 5 is devoted to the case studies and the last section, concludes the paper.

3. Buyers Participation

Electricity retailers are intermediaries because they must purchase energy from suppliers and resell it to the final customers. Retailers must cope with a price and demand risk over a short term time horizon [26]. The main source of these uncertainties is the future pool prices since retailers should purchase electricity from wholesale market with volatile prices and sell it with constant price rates to customers through predetermined contracts. The customers’ actual demand is another source of uncertainty which a retailer should cope with. Therefore, retailers must forecast the spot market prices as well as customers demand. If the higher price spikes have been forecasted, the retailer might bids higher prices to enable more DR capacity. Considering that DR has its maximum capacity in the network, this constraint will limit the amount of purchased DR. In this situation, DR can omit a part of retailer’s financial losses. If a retailer does not deliver the required demand to customers, he/she will be penalized according to the energy not served which is not related to DR contracts. In this study, a new approach is proposed for retailers’ bidding which is based on the load and price forecasted data. Let the retailer purchase electricity from the wholesale market at spot prices and sells the electricity to consumers at a flat rate. Three types of buyers are participated in DR market, which include: retailers, TSOs, and distributors. Here, a new approach is proposed for buyers bidding which is based on the ARIMA forecasting method.

After load and price forecasting, a retailer can participate in DRX market. The aim of a retailer is that, by purchasing a part of required demand from DRX market, instead of energy spot market, decrease his/her costs.

“Fig. 3” shows the proposed procedure of retailer bidding in the DRX market. As it can be seen, the minimum value of the forecasted prices is put as the base value. The aim of a retailer is to minimize his/her costs. To determine the DR demand functions in each hour \( t \), Cartesian coordinates of Fig. 3 are shifted along the horizontal axis so that the vertical axis through from the point which corresponds to the data of hour \( t \). Therefore, the coordination of the considered point which was expressed as \((l_d,p_d)\) in the old Cartesian coordinates, will change to \((0,p_i)\) in the new Cartesian coordinates.

Equation (1) defines the movement equations from all points \((l_d,p_d)\) to the predefined based point \((l_b,p_b)\). \( l_d \) in (1) indicates the absolute value of the difference between the forecasted load in hour \( t \) and the predefined base point and therefore it is equal to \(|l_t-l_b|\). Furthermore, \( P \) and \( L \) in (1), are indicators of the forecasted price and load, respectively. Other variables are defined at the following. It should be noted that “\( l_b \)” in the prior Cartesian coordinates is equivalent to 0 in the new shifted Cartesian coordinates, which should be taken into account in the process of obtaining (1).

\[
P = (p_t-p_i)|l_b| + p_i
\]

where,

\( l_d = |l_t-l_b| \),
\( p_t \): Electricity price in hour \( t \) ($/MWh);
\( p_i \): Electricity price in base hour ($/MWh);
\( l_t \): Amount of load in hour \( t \) (MW);
\( l_i \): Amount of load in the base load point (MW).

All the parameters and variables are schematically shown in “Fig. 3”.

![Fig. 3: Retailer bidding procedure](image)

Equation (1) is obtained from “Fig. 3” through a procedure that is completely defined above and is introduced as the retailers’ bidding function that is based on the forecasted load and price data. Indeed,
retailers will present their aforementioned DR demand functions in a DRX market. Then, DRXO will collect all DR supply and demand curves from both buyer and seller groups and clear them in a common equilibrium point that determine the amount of DR which should be exchanged among the market players. Equation (1) is completely based on the network data and will reasonably satisfy the requirements of DRSs and DRBs.

According to “Fig. 3”, and as shown in (1), the slopes of these linear curves are negative. It illustrates the move from point with higher prices toward a point with minimum price. This linear curve could be treated as retailer demand curve for buying DR from DRX market.

4. Aggregators’ Offering in a DRX Market

This section focuses on supply side of the DRX market. DR sellers want to maximize their profits. In this section, a dynamic approach is developed for aggregators supply function which has been assumed to be a linear curve, as follows:

\[
DRXCP = a_i \times DR_i + b_i \times (1-\theta_i), \quad i=1, \ldots, l
\]

(2)

\(DRXCP\) and \(DR\) are demand response clearing price [in \$/MWh] and the amount of sold DR from \(i^{th}\) customer, respectively. The coefficient \(\theta\) is the “customer type” and represents a customer’s willingness to participate in DR programs. It takes a value between 0 to 1. By increasing in the amount of \(\theta\), the cost of DR decreases because the customer has more willingness to participate in DR. Also, \(a_i\) and \(b_i\) are common coefficients applied to all customers [2].

The amount of traded DR, can be written as a function of \(DRXCP\) as following:

\[
DR_i = \frac{DRXCP - b_i(1-\theta_i)}{a_i}, \quad i=1, \ldots, l
\]

(3)

A balance should be exists between the amount of sold and purchased DR [17]. By considering this constraint (balance between electricity load and supply), we have:

\[
RD = \sum_{i=1}^{l} DR_i = \sum_{i=1}^{l} \frac{DRXCP - b_i(1-\theta_i)}{a_i}
\]

(4)

where,

\(RD\): Required DR

\(l\): Number of customers.

then,

\[
DRXCP = \frac{RD + \sum_{i=1}^{l} b_i(1-\theta_i)}{\sum_{i=1}^{l} a_i}
\]

(5)

Aggregators enroll some customers. If a customer has no willingness to participate in DR, its corresponding \(\theta\) will be equal to 0. Increasing in \(\theta\), shows more willingness of that customer for participating in DR programs.

An approach for \(b_i\) determination for maximizing sellers’ benefit is described in the following. In this approach, each aggregator should maximize his/her benefit in the worst case. The worst condition for aggregators occurs when \(\theta\) tend to 0. In this condition, aggregators have the least capacity to participate in DR, and their profit will be low. By increasing in customers’ willingness, the profit of aggregators will increase, because of their extra capacity to sell in DRX market.

“Fig. 4” illustrates a common cost function of a typical good. In “Fig. 4”, the product “\(x_p\)” is equal to the total income of selling “\(x\)” unit, the hatched area is equal to the cost of providing \(x\) unit, and the shaded area is equal to the net benefit of selling \(x\) unit of this good. Profit function is defined as the total income of selling typical good minus the cost of providing it.

DRSs with high willingness for participating in DRPs have smaller \(b_i\) coefficient in their demand response supply curve. With the order reversed, if DRSs have less willingness, their associated \(b_i\) coefficient will get higher. In this paper, it is assumed that the consumers’ cost functions have quadratic form as the following:

\[
P(s) = DRXCP \times DR - \cos(t, DR), \quad i=1, \ldots, N_{DRS}
\]

(6)

Accurate estimation of consumers cost functions needs accurate investigation and data mining in various energy sectors. Ref [27], investigates the utility function of end-users and proposes some related functions. As it has been described in [27], the utility function can be considered to be quadratic or etc. Participating in DRPs means that customers reduce
their electricity consumption and will lose corresponding utility. Considering this fact, if the revenue of providing DR be less than their pre-existed benefit of electricity consumption, the customers will not be convinced to participate in DRPs. However, other factors can be considered as the customers’ cost functions and they need accurate analysis on various energy sectors which is behind the scope of this paper. It should be noted that this assumption will not affect the generality of this study.

Considering quadratic cost function for the consumers and combining equations (3) and (6) and by substituting \( \theta = 0 \) we will have:

\[
p_f = DRXCP \times \left( \frac{DRXCP - b_i}{a_i} \right) - \left[ \frac{am_i}{2} \left( \frac{DRXCP - b_i}{a_i} \right)^2 + bm_i \left( \frac{DRXCP - b_i}{a_i} \right) \right]
\]

where \( am_i \) and \( bm_i \) are the customers marginal cost function coefficients. It is assumed that \( a_i \) is always equal to \( am_i \) and each seller, changes its supply curve by changing \( b_i \). Each seller can increase its profit by offering higher price offer or larger output amounts by lower price offer. The control variable for each customer is considered to be \( b_i \). By taking the derivative of the profit function with respect to \( b_i \) for customer \( i \), we have:

\[
\begin{bmatrix}
    b_1(k) \\
    b_2(k) \\
    \vdots \\
    b_n(k)
\end{bmatrix} =
\begin{bmatrix}
    \frac{a_1}{a_{11}} \times \frac{1}{a_{11}^2 - 1} & \ldots & \frac{a_1}{a_{1n}} \times \frac{1}{a_{1n}^2 - 1} \\
    \frac{a_2}{a_{21}} \times \frac{1}{a_{21}^2 - 1} & \ldots & \frac{a_2}{a_{2n}} \times \frac{1}{a_{2n}^2 - 1} \\
    \vdots & \ddots & \vdots \\
    \frac{a_n}{a_{n1}} \times \frac{1}{a_{n1}^2 - 1} & \ldots & \frac{a_n}{a_{nn}} \times \frac{1}{a_{nn}^2 - 1}
\end{bmatrix} \begin{bmatrix}
    b_1(k-1) \\
    b_2(k-1) \\
    \vdots \\
    b_n(k-1)
\end{bmatrix}
\]

\[
\begin{bmatrix}
    \frac{b_1}{a_{11}} & 0 & \ldots & 0 \\
    0 & \frac{b_2}{a_{21}} & \ldots & 0 \\
    \vdots & \ddots & \ddots & \vdots \\
    0 & 0 & \ldots & \frac{b_n}{a_{nn}}
\end{bmatrix} \begin{bmatrix}
    b_1(k) \\
    b_2(k) \\
    \vdots \\
    b_n(k)
\end{bmatrix} =
\begin{bmatrix}
    \frac{am_1}{2} \left( \frac{DRXCP - b_i}{a_i} \right)^2 + bm_i \left( \frac{DRXCP - b_i}{a_i} \right)
\end{bmatrix}
\]

and,

\[
DRXCP = \left[ \frac{1}{a_{11}} \frac{1}{a_{21}} \ldots \frac{1}{a_{1n}} \right] \begin{bmatrix}
    b_1(k) \\
    b_2(k) \\
    \vdots \\
    b_n(k)
\end{bmatrix} + \left( \frac{1}{S} \times RD \right)
\]

where,

\[
S = \sum_i \left( \frac{1}{a_i} \right)
\]

The procedure of obtaining (8) and (9) is described in the Appendix section.

5. Numerical Study

To test the proposed approach, the Spain market data in 2002 are used [28]. The results of price and load forecasting by the ARIMA method are shown in “Fig. 5” and “Fig. 6”, respectively. Both Forecasted and real data are shown in these figures.

The per unit daily price error is defined as:

\[
e_{\text{daily price}} = \frac{1}{24} \sum_{t=1}^{24} \left( \frac{p_i^{\text{actual}} - p_i^{\text{forecasted}}}{p_i^{\text{actual}}} \right)
\]

where, \( p_i^{\text{actual}} \) and \( p_i^{\text{forecasted}} \) are the actual and forecasted price data in hour \( t \), respectively. Also, the per unit daily load data is computed as:

\[
e_{\text{daily load}} = \frac{1}{24} \sum_{t=1}^{24} \left( \frac{l_i^{\text{actual}} - l_i^{\text{forecasted}}}{l_i^{\text{actual}}} \right)
\]

where, \( l_i^{\text{actual}} \) and \( l_i^{\text{forecasted}} \) are the actual and forecasted load data in hour \( t \), respectively.

Here, the \( e_{\text{daily price}} \) and \( e_{\text{daily load}} \) values are equal to 9.7 and 1.4 percent, respectively.

Now, consider “Fig. 7”. DR buyers are participated in DRX market as described in section 3. Their demand curves change in each hour depends on the load condition in the network. For the simplicity and without loss of generality, a retailer, a TSO, a distributor, and five DR sellers are assumed here. The bidding curves for all buyers are assumed same as retailer’s demand function. \( \theta \) values are considered as shown in Table I for each sellers. As it can be seen from Table I, sellers 4 and 5, have less willingness to participate in DRX market. But, sellers 1, 2, and 3 have more willingness. Here, 20 percentage of load is assumed as the amount of required DR.
The hourly required DR is less than the sellers' curtailable capacity. So, a competition has been created between sellers to sell DR. This competition occurs in a pool-based market.

$b_i$ coefficients for each seller are obtained using the dynamic approach described in section 4.

DRXO collects both sellers and buyers bidding curves and run the DRX market. The amount of traded DR by each customer is depicted in “Fig. 8”. As it can be seen, each customer wins an amount of DR which is related to its willingness coefficient. Seller 1, cannot participate in DR and Seller 2, can participate in DR only in peak times. Sellers 3, 4, and 5, can participate in DRX market, in most hours. Also, in off-peak

### Table 1: $\theta$ Coefficients for DR sellers

<table>
<thead>
<tr>
<th>DR sellers</th>
<th>$\theta$</th>
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<tbody>
<tr>
<td>Seller 1</td>
<td>0.9</td>
</tr>
<tr>
<td>Seller 2</td>
<td>0.85</td>
</tr>
<tr>
<td>Seller 3</td>
<td>0.78</td>
</tr>
<tr>
<td>Seller 4</td>
<td>0.38</td>
</tr>
<tr>
<td>Seller 5</td>
<td>0.2</td>
</tr>
</tbody>
</table>
intervals, DR cannot be traded between players. This is due to buyers’ unwillingness to enable DR and also less capacity of DR. “Fig. 9”, showing the traded DR price per MWh. Also, the market total benefit during a day is illustrated in “Fig 10”. Pool-based scheme for DR trading, deals DR sellers with multiple buyers in a competitive way and therefore cause more profit for both group of buyers and sellers. Since aggregators bidding strategy is based on the forecasted price and load data, their purchasing functions propose higher prices in peak hours than off-peak periods to purchase from DRX market. This facts, lead to more amount of traded DR among buyers and sellers and therefore will increase the total benefit of participating in DRPs, as it can be seen in “Figs. 8-10”. 

Finally, it can be concluded that the numerical results have satisfied the theoretical concepts of this paper which has been discussed in previous sections as well as applicable viewpoints.

6. Conclusion

This paper addressed a new concept of DR context called DRX. In this study, a novel and systematic approach for DR trading in a market was proposed. The proposed method is completely based on the market condition. Also, DR sellers’ participation was investigated in this paper. A dynamic approach was proposed for sellers’ participation who want to maximize their benefit. Each seller’s supply curve depends on the behavior of other sellers in DRX market in the past times. It also depends on the amount of hourly required DR. The proposed technique is examined using the data of Spain energy market. Studies were conducted to illustrate the benefits of DRX for all players. All simulation results show the efficiency and usefulness of the proposed method.

Appendix

Rewriting (7),

\[ p_f = \frac{DRXCP \times (DRXCP - b_i)}{a_i} - \frac{am \times (DRXCP - b_i)^2}{2a_i} + bm \frac{DRXCP - b_i}{a_i} \]  

(A.1)

DRXCP is a function of \( b_i \), and:

\[ DRX = \frac{b_i - RD + \sum_{i \neq j} b_j - RD}{a_i} + \sum_{i \neq j} \frac{b_j}{a_j} \]  

(A.2)

By taking the derivative of the \( pf_i \) with respect to \( b_i \):

\[ \frac{\partial pf_i}{\partial b_i} = \frac{\partial DRXCP}{\partial b_i} \left( \frac{DRXCP - b_i}{a_i} \right) + \frac{a_i \times \left( \frac{DRXCP - b_i}{a_i} \right) - DRXCP}{a_i} - \frac{am \times \left( \frac{DRXCP - b_i}{a_i} \right) \times DRXCP}{a_i} \]  

(A.3)

Then, by taking the derivative of the \( DRXCP \) with respect to \( b_i \) and putting the result in \( \frac{\partial pf_i}{\partial b_i} \), and by taking the result equal to zero, we conclude:

\[ b_i = \sum_{i \neq j} \left( \frac{a_i}{a_j} \right) - \frac{S}{\sum_{i \neq j} \left( \frac{a_i}{a_j} \right)} \]  

(A.4)

where,

\[ S = \sum_{i \neq j} \left( \frac{1}{a_j} \right) \]

So, equations (8) and (9) are easily derived from (A.4).

References


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